An Overlapping-Generations Model
Of Housing Price Bubbles

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ABSTRACT

In this paper, I introduce an overlapping-generations model of investment where risk-averse agents trade for three periods. When the investments are liquid, the equilibrium is determinate and no bubbles exist. However, with illiquid housing assets, I find the existence of stationary sunspot equilibria which give rise to housing price bubbles. In addition, my results show that with housing in an overlapping-generations economy, nonfundamental volatility results as follows: In order to smooth consumption, some agents endogenously create private claims like asset-backed securities. These claims have multiple price equilibria, and their fluctuating prices cause the prices of housing assets to fluctuate from the asset allocation problem of other agents who must choose between buying real estate directly or investing in asset-backed securities. My model also shows equilibrium idiosyncratic risk bearing in housing markets by certain segments of the population which should be of concern to policy makers. In housing markets, there is a role for government-supplied liquidity and for its active management.

Keywords: overlapping-generations models, housing bubbles, sunspots

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1. INTRODUCTION

Holmstrom and Tirole (1998) define liquidity as the availability of instruments that could be used to transfer wealth across periods. The asset is illiquid if agents' ability to liquidate this asset when desired is restricted. The purpose of this paper is to explore the effects of introducing illiquid housing assets in an overlapping-generations (OG) economy. The model is straightforward: agents “live” and trade for three periods. In the first period, they invest for the second and third periods. If the asset is liquid, the investment could be adjusted in the second period. With illiquid housing assets, the investment is fixed until the third period. This results in indeterminacy[1] and bubbles, where although the economic fundamentals remain constant, prices are volatile.

When fundamental factors do not justify an asset price, then a "bubble" exists. Stiglitz (1990) has pointed out that "if asset prices do not reflect fundamentals well, and if these skewed asset prices have an important effect on asset allocations, then the confidence of economists in the efficiency of market allocations of investment resources is, to say the least, weakened." In the 1970s, the term "bubble" took on another meaning. Is it possible that differences in beliefs generate different dynamic paths - with those beliefs having real effects which are self-fulfilling? That is, are there "boot-strap equilibria" or multiple paths to the economy, all of which are consistent with rational expectations and all of which satisfy the transversality condition? To differentiate terms, these latter phenomena of multiple equilibria were called sunspot equilibria. Here, the actual outcome, undetermined by fundamentals, is determined by speculation based on

[1] Indeterminacy is equated with the existence of non-wandering perfect foresight equilibrium trajectories (i.e. sequences of equilibria over time that never leave some neighborhood in the set of feasible prices).

nonfundamental or extrinsic beliefs called sunspots (that is, the extrinsic beliefs become self-fulfilling). Such nondiversifiable extrinsic risk increases total risk, and thus sunspot equilibria reduce welfare (in a Pareto sense) for risk-averse agents. Also in the 1970s, what was a good model shifted to infinite-horizon models in which a single path satisfied the transversality condition (the saddle-point property), because there was a determinate equilibrium price today and tomorrow, and no bubbles existed.

In this paper, I first show that liquid assets make risk-averse agents better off by eliminating multiple equilibria and thus eliminating speculation based on nonfundamentals. (That is, liquid assets give the economy the saddle-point property as they allow agents to smooth their consumption over time without idiosyncratic risk-bearing.) I then show that with illiquid housing assets, risk-averse agents could be made worse off as the set of economies (agent preferences) that give rise to sunspot equilibria is enlarged. To elaborate, illiquidity of housing assets alone will generate sunspots (even with preference parameters that give determinacy in a pure exchange environment). In addition, my results show that with illiquid housing investment in the economy, the private sector cannot satisfy its own liquidity needs and sunspot equilibria result as follows: In order to smooth consumption, agents endogenously create private claims like asset-backed securities. These claims have multiple price equilibria, and their fluctuating prices cause the prices of capital assets to fluctuate from the asset allocation problem of agents who must choose between buying illiquid real estate directly or investing in these asset-backed securities (the asset demand effect). The resulting fluctuating investments in real estate would affect real returns from strictly concave technologies and make risk-averse agents worse off (in a Pareto sense). Unlike Jarrow and Madan (2000) these asset-backed securities are endogenously created and cause nonfundamental price volatility.[2] My results on the liquidity needs of the private sector extend those of Holmstrom and Tirole (1998) as I include housing assets. In a model with purely financial assets, they find that when there is aggregate uncertainty, the private sector cannot satisfy its own liquidity needs.

In other related literature, Shiller (2001) suggests that "irrational exuberance" is at work in producing elevated housing market levels unjustified by fundamentals. He lists twelve precipitating factors (with amplification mechanisms), as well as cultural and psychological factors which ultimately determine market levels. However, several of his precipitating factors (e.g., tax cuts, baby boom) could represent fundamental shifts in asset demand which would still result in a determinate equilibrium. It could also be argued that none of these factors represents departures from the neoclassical Arrow-Debreu framework per se. Here, I show how one departure - liquidity constraints - generates multiple equilibria which then allows Shiller's cultural and psychological factors to create bubbles from precipitating events.

My model also provides a theoretical basis for empirical studies in fiinance (e.g., Domowitz and El-Gamal, 1999) which suggest that asset illiquidity can lead to the lack of existence of a unique equilibrium.[3] To elaborate, the hypothesis of the ergodicity of prices is rejected for illiquid asset markets, implying that a unique stationary price distribution does not exist and that pricing over a potentially long period following a shock cannot fully reflect market

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[2] Jarrow and Madan (2000) point out that the monetary values (bubble) can fluctuate considerably due to consumer confidence in the economy (a factor not clearly fundamental).

[3] Theoretical finance models of liquidity include Grossman and Miller (1988), and Holmstrom and Tirole (1998). Quantitative models of liquidity-adjusted VaRs include Jarrow and Subramanian (1997, 2001) and Jorion (2000). Empirical papers include Chordia, Roll and Subrahmanyam (2000), Hasbrouck and Seppi (2001) and Huberman and Halka (2001).

fundamentals. This hypothesis cannot be rejected in a demonstrably liquid setting. As noted by Stulz (2000, p.129), "given the role of liquidity in the performance of portfolio insurance in the crash of 1987 and in the collapse of LTCM, it has perhaps revealed itself as the Achilles' heel of finance." Long-Term Capital Management's (LTCM) collapse was so significant that the Federal Reserve felt it necessary to organize LTCM's rescue. Edwards (1999, p.199) reported that "LTCM's vulnerability ... was exacerbated because some of its portfolio consisted of illiquid financial instruments with no ready market, and in some cases it was a substantial holder of these instruments". To unload its position in illiquid assets, LTCM sold them at "firesale" prices, such that by mid-September 1998, LTCM's equity had dropped to $600 million, a loss of more than $4 billion. With the strong reputation of LTCM's general partners (including two Nobel laureates), it could only be concluded that LTCM based its trading strategies on market fundamentals, but, because of illiquidity causing sunspots, the equilibrium predictions of its models never materialized (Jorion, 2000).[4] Jacobs (1999, 2000) also notes the problems from illiquidity in the crash of 1987 and draws parallels to LTCM.

A theoretical physicist (McCauley, 1999) observed that theoretical "econometricians" never bother to discuss relaxation times as they usually assume rapid relaxation to equilibrium. Physicists find this strange, as relaxation time is an important component in dynamical systems analysis. As the illiquid asset could be interpreted as an asset with a longer relaxation time relative to a liquid asset, we would be implicitly incorporating this important component of dynamical systems analysis in financial general equilibrium. In addition, with the illiquid

[4] The "herd behavior" (also noted by Shiller, 2001) is symptomatic of sunspot equilibria which, in game theory, is equivalent to correlated equilibria where the multiple equilibria force the participants to focus on an outcome based on nonfundamental or extrinsic beliefs. The importance of illiquidity in LTCM's collapse has also been noted by The

housing asset, there is always one cohort of traders living which (due to illiquidity or the longer relaxation time) is unable to completely diversify and perfectly smooth consumption. The equilibrium idiosyncratic risk bearing by certain segments of the population is consistent with the empirical evidence (Hayashi, 1985; Zeldes, 1989). The following three sections present the model and formally prove the abovementioned claims. Section V concludes.

2. THE OVERLAPPING-GENERATIONS INVESTMENT ECONOMY

Consider first a stationary overlapping-generations (OG) model of pure exchange (i.e., no capital assets) where economic activity is performed in discrete time. In every period t, N homogeneous agents enter the market or are “born”. Agents “live” and trade for 3 periods. Kehoe and Levine (1983) show that a 3-period-lived agent OG exchange economy with one good is equivalent to two 2-period-lived agents with two goods each period, and use this to prove that both real and nominal steady states may be indeterminate. Since Spear, Srivastava and Woodford (1990) have shown that sunspot equilibria at indeterminate OG steady states require a backward-bending offer curve, I will simply state that whether the 3-period-lived agent OG exchange economy is subject to sunspots depends solely on preferences. In particular, there are open sets of preferences which result in either sunspot or nonsunspot equilibria.

Now introduce a technology utilizing liquid capital assets to such an exchange economy. The agents use their endowments to invest in the first period, invest and consume in the second

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Economist (June 2nd 2001) as "delta-hedging is a game which requires not just vigilance, but also liquid markets" (p.68).

period, and only consume in the last period. There is no loss of generality in assuming that they do not consume in the first period.[5] I will show that liquid apital assets make the OG economy immune to sunspots regardless of preferences and thus improve the welfare of the risk-averse agents.

2.1. The Model.

Agents who enter the market at time t are characterized by their intertemporal utility function $U(c_t^{t+1}, c_t^{t+2})$ defined over non-negative consumption during the second and third periods. I follow Balasko and Shell's (1980) notation for OG models where subscript denotes when the agent entered the market (i.e., the agent's generation) and superscript denotes the time period.

Assumption 2.1. $U(c_t^{t+1}, c_t^{t+2})$ is C^2 and strictly concave in the interior of the consumption set R_+^2. $U(c_t^{t+1}, c_t^{t+2})$ is strictly increasing in c_t^{t+1} and c_t^{t+2} and satisfy boundary (Inada) conditions $U(0, c_t^{t+2}) = U(c_t^{t+1}, 0) = 0$ and first derivatives $U_1(0, c_t^{t+2}) = U_2(c_t^{t+1}, 0) = +\infty$.

In the first period, agents receive an endowment of the consumption good w_t^t which they invest by purchasing liquid real capital assets k_t^t or intergenerational private claims ($w_t^t - k_t^t$). k_t^t earns a gross return of $R^{t+1} k_t^t$ in the second period ($R^{t+1}>1$). In the second period, the agent allocates the liquidated investment returns between consumption c_t^{t+1}, investment in liquid real capital assets k_t^{t+1} (which then yields a gross return of $R^{t+2} k_t^{t+1}$ in the third period, $R^{t+2} > 1$) or

[5] Adding first-period consumption simply adds a column to the Jacobian but the partial derivatives still vanish in

intergenerational private claims. The investments allow the agent to consume c_t^{t+2} in the third period. Defining p^t to be the price of the consumption good in period t (which by construction is also the price of liquid capital assets), each agent's problem can be written as:

$$\max_{c_t^{t+1}, c_t^{t+2}, k_t^t, k_t^{t+1}} U(c_t^{t+1}, c_t^{t+2}) \tag{2.1}$$

Subject to the budget constraint with multiplier λ_t:

$$p^t k_t^t + p^{t+1}(c_t^{t+1} + k_t^{t+1}) + p^{t+2} c_t^{t+2} \leq p^t w_t^t + p^{t+1} R^{t+1} k_t^t + p^{t+2} R^{t+2} k_t^{t+1}$$

which with the assumption of non-satiation can be rewritten as:

$$p^t(w_t^t - k_t^t) + p^{t+1}(R^{t+1} k_t^t - c_t^{t+1} - k_t^{t+1}) + p^{t+2}(R^{t+2} k_t^{t+1} - c_t^{t+2}) = 0 \tag{2.2}$$

Notice that k_t^t, c_t^{t+2} and k_t^{t+1} are set independently as they determine the excess demands of each generation (the negative of which are the amounts of intergenerational private claims held). The first-order conditions (FOCs) for the agent's problem are, with respect to c_t^{t+1}, c_t^{t+2}, k_t^t and k_t^{t+1} respectively:

$$U_{1,t} - p^{t+1}\lambda_t = 0 \tag{2.3}$$

the Proof of Proposition 2.4.

$$U_{2,t} - p^{t+2}\lambda_t = 0 \quad (2.4)$$

$$\lambda_t[p^{t+1}R^{t+1} - p^t] = 0 \quad (2.5)$$

$$\lambda_t[p^{t+2}R^{t+2} - p^{t+1}] = 0 \quad (2.6)$$

(2.3) implies that $\lambda_t = U_{1,t} / p^{t+1} > 0$ and so (2.4), (2.5) and (2.6) can be rewritten as follows:

$$- p^{t+1}U_{2,t} + p^{t+2} U_{1,t} = 0 \quad (2.7)$$

$$R^{t+1} = p^t / p^{t+1} \quad (2.8)$$

$$R^{t+2} = p^{t+1} / p^{t+2} \quad (2.9)$$

Assume an intertemporally separable constant returns to scale (CRS) technology with well-defined marginal products each period allowing investors to have at all times that combination of inputs which they would wish to have at the given prices (i.e. the technology admits variable proportions). Due to the CRS technology and perfectly substitutable liquid real capital assets, the returns from the technology may be written as:

$$G(k^t_t, k^t_{t-1}) = f(k^t_t) + g(k^t_{t-1}) \quad (2.10)$$

Assumption 2.2. f (g) is C^2, positive, strictly increasing and strictly concave in k_t^t (k_{t-1}^t) and there exists a $\bar{k}$ such that f($\bar{k}$) = $\bar{k}$ and g($\bar{k}$) = $\bar{k}$. Let **K** = [0, $\bar{k}$] and for all t, k_t^t € int **K** and k_{t-1}^t € int **K**. The functions f and g are such that f(0)=g(0)=0, f'(0)=g'(0)=+∞ and f'($\bar{k}$)<1 and g'($\bar{k}$)<1.

Assumption 2.2 admits a level of capital $\hat{k}$ where f'($\hat{k}$)=1 and g'($\hat{k}$)=1. Restricting our attention to efficient real steady states in this section is then equivalent to restricting k_t^t, k_{t-1}^t € int [0, $\hat{k}$] (i.e., $R^{t+1} > 1$ so that the liquid capital assets yield strictly positive net rates of return). Dynamics near inefficient real states have been studied by Jullien (1988) in a two-period OG model where he assumes that $\hat{k}$ is a lower bound instead of an upper bound for capital. That reasonable parameters consistent with f'>1 or f'<1 in equilibrium exist has been shown by Blanchard and Fisher (1989, p.104).

Competition in the market for liquid real capital assets ensures that the asset is paid its marginal product. Hence for agents who enter the market in period t:

$$R^{t+1} = f'(k_t^t) \tag{2.11}$$

Then (2.11) and (2.8) imply:

$$- p^t + p^{t+1} f'(k_t^t) = 0 \tag{2.12}$$

Similarly:

$$R^{t+1} = g'(k^{t}_{t-1}) \tag{2.13}$$

Since (2.13) must hold for all generations, (2.9) implies:

$$- p^{t+1} + p^{t+2} g'(k^{t+1}_{t}) = 0 \tag{2.14}$$

The demand equations for the agent who enter the market in t, which involve (2.7), (2.11) and (2.13) as well as the agent's budget constraint (2.2), can be solved for (c^{t+1}_{t}, c^{t+2}_{t}, k^{t}_{t}, k^{t+1}_{t}) being continuously differentiable with respect to (p^{t},p^{t+1},p^{t+2}) if the Jacobian J is non-singular, where:

$$J = - p^{t+1}f'' p^{t+2}g''[(p^{t+2})^2U_{11} - 2p^{t+1} p^{t+2}U_{12} + (p^{t+1})^2U_{22}] \tag{2.15}$$

With $\{p^{t}\}>0$, Assumptions 2.1 and 2.2 guarantee that J is non-degenerate. To see this, strict concavity of U in Assumption 2.1 implies that for any B≠0, $B^{T}D^{2}UB<0$, where $D^{2}U$ is the matrix of second derivatives (Hessian) of U. The term in braces is simply the quadratic form of $B^{T}D^{2}UB$, where:

$$B^{T} = [\ p^{t+2}\ \ -p^{t+1}\ ] \tag{2.16}$$

The strict concavity of f and g in Assumption 2.2 ensures f’’ and g’’ are strictly negative, which implies that J>0. The Implicit Function Theorem then allows me to compute the partial derivatives of the demand functions for c_t^{t+1}, c_t^{t+2}, k_t^t and k_t^{t+1}. Finally, to make the economy self-sustaining, assume the endowment to agents in the first period w_t^t is simply the excess return from a competitive capital market:

$$w_t^t = f(k_{t-1}^{t-1}) - f'(k_{t-1}^{t-1})\,k_{t-1}^{t-1} + g(k_{t-2}^{t-1}) - g'(k_{t-2}^{t-1})\,k_{t-2}^{t-1} \tag{2.17}$$

The feasibility or equilibrium condition in period t is:

$$(w_t^t - k_t^t) + (f'(k_{t-1}^{t-1})\,k_{t-1}^{t-1} - c_{t-1}^t - k_{t-1}^t) + (g'(k_{t-2}^{t-1})\,k_{t-2}^{t-1} - c_{t-2}^t) = 0 \tag{2.18}$$

Substituting for w_t^t from (2.17) implies:

$$f(k_{t-1}^{t-1}) + g(k_{t-2}^{t-1}) - k_t^t - c_{t-1}^t - k_{t-1}^t - c_{t-2}^t = 0 \tag{2.19}$$

The equilibrium condition (2.19) then depends on prices as follows:

$$\begin{aligned} & f(k_{t-1}^{t-1}(p^{t-1},p^t,p^{t+1})) + g(k_{t-2}^{t-1}(p^{t-2},p^{t-1},p^t)) \\ & - k_t^t(p^t,p^{t+1},p^{t+2}) - c_{t-1}^t(p^{t-1},p^t,p^{t+1}) - k_{t-1}^t(p^{t-1},p^t,p^{t+1}) - c_{t-2}^t(p^{t-2},p^{t-1},p^t) = 0 \end{aligned} \tag{2.20}$$

2.2. Steady-State Analysis.

I first show that a steady-state where real interest rates are positive exists. Next, I show that the strictly positive net return from the liquid real capital asset results in agents always choosing to have zero excess demands for private claims. That is, private claims are dominated by the liquid asset and would not exist. Then I demonstrate how the liquid asset reduces the 3-period-lived agent one-good OG model to a 2-period-lived agent one-good OG model, that is, there is a collapse in dimensionality even when preferences are not inter-temporally separable. This allows me to prove that under Assumptions 2.1 and 2.2, the positive real interest rate steady-state of a 3-period-lived agent OG model with jelly capital is determinate.

Following Kehoe and Levine (1983, 1984, 1985), a steady state is a relative price vector p and a price evolution factor β such that $p^t = \beta^t p$ is an equilibrium of the economy for all t. The steady state real rate of interest is then 1/β - 1. At the steady state, (2.12), (2.14) and (2.7) imply that for all t:

$$f'(k_t^t) = p^t / p^{t+1} = 1/\beta \qquad (2.21)$$

$$g'(k_t^{t+1}) = p^{t+1} / p^{t+2} = 1/\beta \qquad (2.22)$$

$$U_{2,t} = \beta U_{1,t} \qquad (2.23)$$

At positive real interest rate steady-states, β<1 since f'>1 for all $k_t^t < \hat{k}$ and g'>1 for all $k_t^{t+1} < \hat{k}$. With demand functions being homogeneous of degree zero (HD0) in prices, the equilibrium condition at the steady-state (2.20) is:

$$f(k_{t-1}^{t-1}(p, \beta p, \beta^2 p)) + g(k_{t-2}^{t-1}(p, \beta p, \beta^2 p)) - k_t^t(p, \beta p, \beta^2 p) - c_{t-1}^t(p, \beta p, \beta^2 p) - k_{t-1}^t(p, \beta p, \beta^2 p) - c_{t-2}^t(p, \beta p, \beta^2 p) = 0 \quad (2.24)$$

With every generation having identical preferences, their demands must be equivalent if they face the same set of prices. Therefore:

$$f(k_t^t(p, \beta p, \beta^2 p)) + g(k_t^{t+1}(p, \beta p, \beta^2 p)) - k_t^t(p, \beta p, \beta^2 p) - c_t^{t+1}(p, \beta p, \beta^2 p) - k_t^{t+1}(p, \beta p, \beta^2 p) - c_t^{t+2}(p, \beta p, \beta^2 p) = 0 \quad (2.25)$$

Proposition 2.3. *(Existence of Equilibrium)* Under Assumptions 2.1 and 2.2, there exists a steady state equilibrium price p and evolution factor β<1 such that $(p, \beta p, \beta^2 p) \in R_+^3$, and satisfies (2.24) and (2.25).

Proof of Proposition 2.3. See Appendix.[6] ■

If real rates of return are positive, then holding liquid assets dominate holding intergenerational private claims (which yield no real return). Denoting the excess demand of

[6] Appendix available from the author.

agents who enter the market in period t by μ_t, (2.2) and (2.25) imply that $\mu_t = \mu_{t-1} = \mu_{t-2} = 0$. The next proposition shows that liquid assets in a variable proportions technology reduce the 3-period-lived agent one-good OG model to a 2-period-lived agent one-good OG model at the real steady-states.

Proposition 2.4. *(Reduction of Dimensionality of OG Economy)* Under Assumptions 2.1 and 2.2, the dynamics in an open neighborhood of the real steady-states ($\beta \neq 1$) of the one-good, 3-period-lived agent OG model with liquid real capital assets are identical to the dynamics in an open neighborhood of the real steady-states of a one-good, 2-period-lived agent pure exchange OG model.

Proof of Proposition 2.4. See Appendix.[7] ■

Corollary 2.5. *(Determinacy of Equilibrium)* Under Assumptions 2.1 and 2.2, perfect foresight equilibria cannot be indeterminate and are in fact immune to sunspots near the efficient real steady-states of the 3-period-lived agent OG model with liquid real capital assets.

Proof of Corollary 2.5. Kehoe and Levine (1985, pp. 446-447) showed that a one-good two-period lived OG model cannot be indeterminate near the real steady-states. Since $\beta<1$, no path with nominal initial conditions can ever approach the real steady states with the eigenvalues being 1 and $1/\beta$. Finally, determinacy of equilibria is sufficient to rule out sunspot equilibria. ■

[7] Appendix available from the author.

Even when we allow for negative real interest rate steady states ($\beta>1$), the collapse of dimensionality is generic and the OG economy is locally determinate and immune to sunspots. However, if $\beta>1$, Kehoe and Levine (1985) say that the steady-state is globally indeterminate. The existence of liquid real capital assets which dominate intergenerational private claims (in rate-of-return) is necessary to get the row or column of zeros in the partial derivative matrix and the collapse in dimensionality resulting in determinacy. Benveniste, Capozza and Seguin (2001) find that liquid individual housing properties adds 16% to their value. Such liquidity could be achieved if the marketing period was made shorter and more certain (Lin and Vandell, 2007). That liquid real housing assets result in determinate housing prices suggest faster mean reversion, and Capozza, Hendershott and Mack (2004) find evidence of this. Without liquid assets, intergenerational private claims would exist and nominal steady state equilibria would be indeterminate (see Grandmont (1985) and Jullien (1988)) as the next section shows.

3. THE OVERLAPPING-GENERATIONS ECONOMY WITH ILLIQUID HOUSING INVESTMENT

Consider now the model in Section 2.1 but with illiquid housing investment instead. The difference affects agents in the second period as the housing investment's gross return structure is fixed for two periods. That is, agents can invest or divest in housing only in the first and third periods. Since net returns of housing assets are strictly positive only over two periods (and we allow negative returns over one period), intergenerational private claims endogenously arise.

However, these private claims (like asset-backed securities, for example) create indeterminacy and increase the likelihood of sunspot equilibria, as I will now formally demonstrate.

3.1. The Set-Up.

Agent preferences are defined as per Assumption 2.1. In the first period, agents receive an endowment of the consumption good w_t^t which they invest in illiquid housing assets x_t^t or intergenerational private claims ($w_t^t - x_t^t$). x_t^t provides a return of $R_t^{t+1} x_t^t$ in the second period for the agent to consume c_t^{t+1}. When excess demand of the agent in the second period is positive (i.e., $c_t^{t+1} - R_t^{t+1} x_t^t > 0$), private claims backed by third period real returns could be shorted to smooth consumption. The housing asset yields a gross return $R_t^{t+2} x_t^t$ in the third period, which allows the agent to consume c_t^{t+1} and to repay the shorted private claims. Noting that the subscript in the gross return variable R denotes the vintage of the housing stock and defining p^t to be the price of the consumption good in period t, each agent's problem can be written as:

$$\max_{c_t^{t+1}, c_t^{t+2}, x_t^t} U(c_t^{t+1}, c_t^{t+2}) \tag{3.1}$$

subject to the budget constraint with multiplier λ_t:

$$p^t x_t^t + p^{t+1} c_t^{t+1} + p^{t+2} c_t^{t+2} \leq p^t w_t^t + p^{t+1} R_t^{t+1} x_t^t + p^{t+2} R_t^{t+2} x_t^t$$

which with the assumption of non-satiation can be rewritten as:

$$p^t(w_t^t - x_t^t) + p^{t+1}(R_t^{t+1} x_t^t - c_t^{t+1}) + p^{t+2}(R_t^{t+2} x_t^t - c_t^{t+2}) = 0 \qquad (3.2)$$

The first-order conditions (FOCs) for the agent's problem are, with respect to c_t^{t+1}, c_t^{t+2} and x_t^t respectively:

$$U_{1,t} - p^{t+1}\lambda_t = 0 \qquad (3.3)$$

$$U_{2,t} - p^{t+2}\lambda_t = 0 \qquad (3.4)$$

$$\lambda_t[p^{t+2} R_t^{t+2} + p^{t+1} R_t^{t+1} - p^t] = 0 \qquad (3.5)$$

(3.3) implies that $\lambda_t = U_{1,t} / p^{t+1} > 0$ and so (3.4) and (3.5) can be rewritten as follows:

$$- p^{t+1}U_{2,t} + p^{t+2} U_{1,t} = 0 \qquad (3.6)$$

$$p^t = p^{t+1} R_t^{t+1} + p^{t+2} R_t^{t+2} \qquad (3.7)$$

Next, imagine a housing developer in period t with an existence of three periods and using a technology which is not *ex-post* variable. The technology yields a gross output of $f(x_t^t)$ in period t+1 and $g(x_t^t)$ in period t+2. $f(x_t^t)$ and $g(x_t^t)$ satisfy the following assumption:

Assumption 3.1. f and g are CRS, C^2, positive and strictly concave in x_t^t and there exists a $\bar{x}$ such that $f(\bar{x}) = \bar{x}$ and $g(\bar{x}) = \bar{x}$. Let $\mathbf{X} = [0, \bar{x}]$ and for all t, x_t^t € int **X**. The functions f and g are such that f(0)=g(0)=0, f'(0)=g'(0)=+∞ and $f'(\bar{x})<1$ and $g'(\bar{x})<1$.

Assumption 3.1 admits a level of capital $\hat{x}$ where $f'(\hat{x})+\beta' g(\hat{x})=1$ for any β<1. Restricting our attention to efficient real steady states is then equivalent to restricting x_t^t € int $[0, \hat{x}]$ where the real capital assets yield strictly positive net rates of return over two periods. The developer's optimization problem is:

$$\max_{x_t^t} \; p^{t+1}[f(x_t^t) - R_t^{t+1} x_t^t] + p^{t+2}[g(x_t^t) - R_t^{t+2} x_t^t] \qquad (3.8)$$

where R_t^{t+1} (R_t^{t+2}) is what the developer tells investors housing returns will be in period t+1 (t+2). The FOC for (3.8) is then:

$$p^{t+1}[f'(x_t^t) - R_t^{t+1}] + p^{t+2}[g'(x_t^t) - R_t^{t+2}] = 0 \qquad (3.9)$$

Since (3.9) must hold for all t, (3.7) implies:

$$-p^t + p^{t+1}f'(x_t^t) + p^{t+2}g'(x_t^t) = 0 \qquad (3.10)$$

The demand equations for the agent who enter the market in t, which involve (3.6) and (3.10) as well as the agent's budget constraint (3.2), can be solved for (c_t^{t+1}, c_t^{t+2}, x_t^t) being continuously differentiable with respect to (p^t,p^{t+1},p^{t+2}) if the Jacobian J is non-singular, where:

$$J = -(p^{t+1}f'' + p^{t+2}g'')[(p^{t+2})^2U_{11} - 2p^{t+1}p^{t+2}U_{12} + (p^{t+1})^2U_{22}] \qquad (3.11)$$

With $\{p^t\}>0$, Assumptions 2.1 and 3.1 guarantee that J is non-degenerate. With J<0, the Implicit Function Theorem allows me to compute the partial derivatives of the demand functions for c_t^{t+1}, c_t^{t+2} and x_t^t. To make the economy self-sufficient as in the previous section, let:

$$w_t^t = f(x_{t-1}^{t-1}) - f'(x_{t-1}^{t-1})x_{t-1}^{t-1} + g(x_{t-2}^{t-2}) - g'(x_{t-2}^{t-2})x_{t-2}^{t-2} \qquad (3.12)$$

The feasibility or equilibrium condition in period t is:

$$(w_t^t - x_t^t) + (f'(x_{t-1}^{t-1})x_{t-1}^{t-1} - c_{t-1}^t) + (g'(x_{t-2}^{t-2})x_{t-2}^{t-2} - c_{t-2}^t) = 0 \qquad (3.13)$$

Substituting for w_t^t from (3.12) implies:

$$f(x_{t-1}^{t-1}) + g(x_{t-2}^{t-2}) - x_t^t - c_{t-1}^t - c_{t-2}^t = 0 \qquad (3.14)$$

The equilibrium condition (3.14) then depends on prices as follows:

$$f(x_{t-1}^{t-1}(p^{t-1},p^{t},p^{t+1})) + g(x_{t-2}^{t-2}(p^{t-2},p^{t-1},p^{t}))$$

$$- x_{t}^{t}(p^{t},p^{t+1},p^{t+2}) - c_{t-1}^{t}(p^{t-1},p^{t},p^{t+1}) - c_{t-2}^{t}(p^{t-2},p^{t-1},p^{t}) = 0 \quad (3.15)$$

3.2. Real Steady-State Analysis.

I first derive some steady-state restrictions. Then I show that there exists a real steady-state equilibrium. Finally, I show that there is one dimension of indeterminacy with illiquid housing assets for an open set of economies near positive real interest rate ($\beta<1$) steady states. *This set of economies is larger than the set of indeterminate pure exchange OG economies.* A steady state is a relative price vector p and a price evolution factor β such that $p^t=\beta^t p$ is an equilibrium of the economy for all t. At the steady state, (3.6) and (3.10) imply that, for all t:

$$U_{2,t} = \beta U_{1,t} \quad (3.16)$$

$$f'(x_t^t) + \beta g'(x_t^t) = 1/\beta \quad (3.17)$$

With demand functions being homogeneous of degree zero (HD0) in prices, the equilibrium condition at the steady-state (3.15) can be written as:

$$f(x_{t-1}^{t-1}(p, \beta p, \beta^2 p)) + g(x_{t-2}^{t-2}(p, \beta p, \beta^2 p))$$

$$- x_{t}^{t}(p, \beta p, \beta^2 p) - c_{t-1}^{t}(p, \beta p, \beta^2 p) - c_{t-2}^{t}(p, \beta p, \beta^2 p) = 0 \quad (3.18)$$

With every generation having identical utility and production functions, their demands must be equivalent if they face the same set of prices. Thus:

$$f(x_t^t(p, \beta p, \beta^2 p)) + g(x_t^t(p, \beta p, \beta^2 p))$$

$$- x_t^t(p, \beta p, \beta^2 p) - c_t^{t+1}(p, \beta p, \beta^2 p) - c_t^{t+2}(p, \beta p, \beta^2 p) = 0 \qquad (3.19)$$

Proposition 3.2. *(Existence of Equilibrium)* Under Assumptions 2.1 and 3.1, there exists a steady state equilibrium price p and evolution factor $\beta<1$ such that $(p, \beta p, \beta^2 p)$ € R_+^3, and satisfies (3.18) and (3.19).

Proof of Proposition 3.2. See Appendix.[8] ■

Proposition 3.3. *(Indeterminacy of Steady State Equilibria)* Under Assumptions 2.1 and 3.1 and for a non-empty, open set of OG economies with illiquid real capital assets, perfect foresight equilibria have a one-dimensional indeterminacy near real steady-states where $\beta<1$.

Proof of Proposition 3.3. See Appendix.[8] ■

Corollary 3.4. The non-empty, open set of indeterminate OG economies with illiquid real capital assets is larger than the non-empty open set of indeterminate pure-exchange OG economies.

Proof of Corollary 3.4. See Appendix.[8] ■

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[8] Appendix available from the author.

The indeterminate OG economies have steady-states β where g' is sufficiently large and positive, f' sufficiently small or negative and -f" sufficiently large. How do we interpret these restrictions? g' large means that agents get a high return from the housing asset after two periods. f' small or negative means that agents get a low or negative return after one period, that is, the housing asset yields almost nothing in the second period and could not even be liquidated early without a severe price discount.[9] f" sufficiently large means that the low return f' in the second period could quickly turn negative. In short, *illiquidity is sufficient for indeterminacy.* Liquidity plummeting during "down markets" is consistent with Chordia, Roll and Subrahmanyam's (2001) evidence. Since the housing asset could not be liquidated and its one-period return is low (or negative), some generations of agents in the second period have to trade private financial claims to smooth consumption. This is akin to selling private asset-backed securities in the second period, with the securities backed by the returns from the housing asset in the third period. These private asset-backed securities or financial claims do *not* make the equilibrium determinate, unlike the liquid real assets of the previous section. In the next section, I show how indeterminacy of equilibria with illiquid housing assets and private claims lead to sunspot equilibria or nonfundamental price volatility.

4. STATIONARY SUNSPOT EQUILIBRIA WITH ILLIQUID HOUSING ASSETS

[9] The return could be the rents from housing. Indeterminacy arises in the housing market when housing assets could not be sold nor rented in the second period.

With illiquid housing assets in the previous section, the resulting indeterminacy typically implies the existence of stationary sunspot equilibria of the Spear, Srivastava and Woodford (1990) type. Since sunspot equilibria are typically inefficient, they make risk-averse agents worse off. In this section, I prove the existence of stationary sunspot equilibria in the illiquid housing assets OG economy. The sunspots could be Shiller's (2001) precipitating or cultural or psychological factors which shifts the OG economy between different multiple equilibria each period (where the multiple equilibria arise from illiquidity).

The stochastic optimization problem of each agent can be written as:

$$\max_{c_t^{t+1},\, c_t^{t+2},\, x_t^t} E_t U(c_t^{t+1}, c_t^{t+2}) \tag{4.1}$$

subject to per period budget constraints. With no discounting, the developer's stochastic optimization problem is:

$$\max_{x_t^t} E_t\{p^{t+1}[f(x_t^t) - R_t^{t+1} x_t^t] + p^{t+2}[g(x_t^t) - R_t^{t+2} x_t^t]\} \tag{4.2}$$

where E_t is the conditional expectation given information on prices at period t.

(4.1) and (4.2) imply that the optimization problem for the agent and developer in period t require them to make forecasts of prices p^{t+1} and p^{t+2}. Therefore, the sunspot shocks for both periods t+1 and t+2 must matter jointly. I permit this by letting ξ^t denote the R-valued independently and identically distributed (i.i.d.) mean zero sunspot variable (shock) in period t. Let $\xi(\xi^{t+1},\xi^{t+2})$ be an R-valued measurable function of the sunspot shocks in t+1 and t+2 which enters the optimization problems of agents and developers in period t. For simplicity, let $\xi(\xi^{t+1},\xi^{t+2})$ be drawn from a stationary, i.i.d. distribution ψ such that[10]:

$$\int \xi \, d\psi(\xi) = 0 \tag{4.3}$$

Following the bootstrapping technique of Farmer and Woodford (1997) and Spear, Srivastava and Woodford (1990), construct a large family of equilibria by replacing (3.10), (3.6) and (3.2) with:

$$-p^{t} + p^{t+1}f'(x_t^t) + p^{t+2}g'(x_t^t) - \xi(\xi^{t+1},\xi^{t+2}) = 0 \tag{4.4}$$

$$- p^{t+1}U_{2,t} + p^{t+2}\, U_{1,t} - \xi(\xi^{t+1},\xi^{t+2}) = 0 \tag{4.5}$$

$$p^{t}(w_t^t - x_t^t) + p^{t+1}(f'(x_t^t)\,x_t^t - c_t^{t+1}) + p^{t+2}(g'(x_t^t)\,x_t^t - c_t^{t+2}) - \xi(\xi^{t+1},\xi^{t+2}) = 0 \tag{4.6}$$

With $\xi(\xi^{t+1},\xi^{t+2})$ exogenous and thus independent of $(c_t^{t+1},\ c_t^{t+2},\ x_t^t)$, (3.11) shows that (4.4), (4.5) and (4.6) can be solved for $(c_t^{t+1},\ c_t^{t+2},\ x_t^t)$ uniquely in terms of $(p^t,p^{t+1},p^{t+2},\xi)$ as long as supp ξ € $\eta(0)$, where $\eta(0)$ is a sufficiently small neighborhood of 0. Lagging this result two periods, I can write:

[10] The set of i.i.d. functions ξ is non-empty. $\xi(\xi^{t+1},\xi^{t+2}) = \xi^{t+2}$ is i.i.d..

$$c_{t-2}^{t} = C(p^{t-2},p^{t-1},p^{t},\xi) \quad (4.7)$$

Working through the equilibrium condition, (4.7) implies:

$$x_{t}^{t} = X(p^{t-2},p^{t-1},p^{t},\xi) \quad (4.8)$$

I solve for p^{t+2} in (4.4) to yield:

$$p^{t+2} = [p^{t} - p^{t+1}f'(x_{t}^{t}) + \xi(\xi^{t+1},\xi^{t+2})] / g'(x_{t}^{t}) \quad (4.9)$$

or more generally that:

$$p^{t+2} = g^{*}(p^{t+1}, p^{t}, x_{t}^{t}, \xi) \quad (4.10)$$

since f is a function of x_{t}^{t}. Now substituting for x_{t}^{t} from (4.8) yields:

$$p^{t+2} = g^{*}(p^{t+1}, p^{t}, p^{t-1}, p^{t-2}, \xi) \quad (4.11)$$

which is then the forecast function used by agents. By construction, these forecasts yield demand functions which solve the stochastic FOCs (4.4)-(4.6) for expected utility maximization. (4.11) is also the equilibrium price evolution which clears the market clearing condition (3.14) by construction. Furthermore, g* is clearly a continuous function of its arguments. Using the definition of Spear (1988) for rational expectations forecasts and rational expectations equilibria, I have the following two propositions:

Proposition 4.1. *(Existence of Stationary Sunspots)* There exists a non-empty, open set of OG economies with illiquid real capital assets exhibiting non-trivial stationary sunspot equilibria.

Proof of Proposition 4.1. With Propositions 3.2 and 3.3 and the above discussion, it remains to show that there exists an invariant measure for the forecast function $p^{t+2} = g^*(p^{t+1}, p^t, p^{t-1}, p^{t-2}, \xi)$. First, let $\xi(\xi^{t+1},\xi^{t+2}) = \xi_t$. Now for any continuous function h of (p^{t+2}, ξ_t), define for $\xi_t \in \zeta$, ξ_t i.i.d.,

$$P\, h(p^{t+1}, p^t, p^{t-1}, p^{t-2}, \xi_{t-1}) = \int_{\zeta} h[g^*(p^{t+1}, p^t, p^{t-1}, p^{t-2}, \xi_{t-1}, \xi_t), \xi_t]\, d\psi(\xi_t) \qquad (4.12)$$

Since g* is a continuous function of its arguments, by the Implicit Function Theorem, the transition operator P plainly takes continuous functions into continuous functions. Hence by Rosenblatt's (1959) Theorem, there exists an invariant distribution for the price formation process $(p^{t+1},p^t,p^{t-1},p^{t-2},\xi_{t-1})$. This distribution, together with the forecast function g*, constitutes a stationary rational expectations equilibrium (REE). Since the random variable ξ_t is nondegenerate, the equilibrium is stochastically nontrivial. ■

Proposition 4.2. The non-empty, open set of OG sunspot economies with liquid real capital assets is larger than the non-empty, open set of pure-exchange OG sunspot economies.

Proof of Proposition 4.2. It follows from the proof of Proposition 3.3 which shows the stability of real steady-state equilibria that is sufficient for the existence of stationary sunspot equilibria (Spear, Srivastava and Woodford, 1990), Corollary 3.4 which shows that the set of stable OG economies is larger with illiquid real capital assets than for pure-exchange, and Proposition 4.1 above. ■

Sunspot equilibria result in price volatility as follows: the illiquid housing asset forces some generations of agents to short private claims in the second period to smooth consumption. Because of indeterminacy and sunspots, the prices and returns of these claims vary stochastically even though fundamentals are deterministic. The fluctuating prices of private claims cause the prices of housing assets to fluctuate from the asset allocation problem of agents who must choose between buying illiquid real estate directly or investing in private claims like asset-backed securities (the asset demand effect). Due to excess or nonfundamental volatility of real returns, risk-averse agents are made worse off (in a Pareto sense).

5. CONCLUSION

In a one-good, 3-period-“lived” agent overlapping-generations model, I have shown how typically indeterminate positive real interest rate steady states become determinate and immune to sunspots with liquid assets which allow agents to perfectly smooth their consumption over time. I have then shown how indeterminacy occurs robustly with illiquid housing assets. In addition, illiquidity of housing assets alone will generate bubbles and sunspots (even with preferences that give determinacy in an exchange environment). Thus the set of sunspot economies is in fact larger than in the case of pure exchange. These results suggest that an economy with illiquid assets has higher volatility than an economy with liquid assets. That liquidity is negatively correlated with volatility has been documented empirically by Huberman and Halka (2001).

With this model, it is evident how Shiller's precipitating, cultural and psychological factors create "irrational exuberance" in dynamic general equilibrium (which is simply due to liquidity constraints). A theoretical basis to show how illiquidity creates nonunique equilibria leading to nonfundamental equilibria has also been provided. An extension of this paper involves some idiosyncratic technological shocks to output. In this case, our model can be viewed as an extreme case where these shocks are contained in a sufficiently small neighborhood of the non-stochastic technology. In a sense, the illiquid housing asset model suggests that sunspots could amplify certain small aggregate technological shocks to generate large output fluctuations. Because the indeterminacy results from liquidity constraints and not external increasing returns, these shocks would not create a unique equilibrium, unlike Frankel and Pauzner (2000).

In conclusion, my results should be of concern to regulators and policy makers. In economies with illiquid real assets (like housing), there is a role for government intervention (see Shiller, 2001, p.230 and p.288 footnote 29) and for government-supplied liquidity and for its active management (see Holmstrom and Tirole, 1998). In my model, due to liquidity constraints, there always exists “unlucky” generations which are forced to short private claims backed by real asset returns in the future in order to smooth consumption. The “lucky” generations would purchase claims in the first period and redeem the claims in the second period to consume. Thus the “lucky” generation would have no need to short claims in the second period. The alternating “unlucky” generations are forced to invest all their first period endowments in the illiquid asset and short claims to the “lucky” generations in the second period in order to consume. In the context of housing, the “unlucky” generations put all their endowments into housing in the first period and suffer significant idiosyncratic risk.

In the housing market, the unlucky generations could represent investors and homeowners who are unable to rent or sell their homes and whose mortgages are approaching junk status. This is consistent with empirical evidence showing that certain segments of the population suffer more from nonfundamental asset or housing price volatility than others. Still, these mortgages could be backed by future returns from housing (either through sale or rental income) and these mortgages could be packaged and sold to other investors as private label “asset-backed securities”. Hendershott, Hendershott and Shilling (2010) report that private asset-backed securities (ABS) increased eightfold during the U.S. housing bubble form 2001 to 2006. The monthly spread over 10-year U.S. Treasuries was volatile, going from 500 basis points in 2000 down to 100 basis points in 2004 and then skyrocketing to 700 basis points in 2008 and peaking at 2,100 basis points in 2009. This private label asset-backed securitization collapsed in 2009 and is currently nonexistent. Without private asset-backed securities keeping junk mortgages afloat, the “unlucky” generation of investors and homeowners face foreclosure.

In conclusion, there will always be idiosyncratic risk-bearing in practice, which suggests "a good outcome can be achieved by designing better forms of social insurance and creating better financial institutions to allow the real risks to be managed more effectively" (Shiller, 2001, p.233). One may also make a case for capital taxation to reduce after-tax asset return volatility (Cadenillas and Pliska, 1999) or to achieve more equitable outcomes in society.

6. REFERENCES

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