Minority Interests in Real Estate Limited Partnerships and Discounts for Illiquidity and Lack of Control

James D. Shilling†
DePaul University

Draft: September 2013

Abstract

This paper develops a search model that determines the combined discount for lack of liquidity and lack of control for minority interests in real estate limited partnerships. The model is calibrated to data on multifamily residential apartment buildings. The combined discounts computed from the model are compared to those observed from the data. With realistic parameter values, the model is able to generate discounts for lack of liquidity and lack of control that are consistent with the empirical literature.

Keywords: Search Theory, Illiquidity, Ownership Structure, Market Valuation

JEL Classifications: D83, G32

†DePaul University, 1 East Jackson Boulevard, Chicago, IL 60604, Email: shilling@depaul.edu

1 Introduction

Extremely high discounts play a central role in the valuation of minority-limited partnership interests in real estate limited partnerships. Indeed, selling a minority interest in real estate limited partnership can be difficult in part because there is no secondary market for minority shares and partly because minority interests represent the interest of non-controlling unitholders in the limited partnership. Not only that, but once a potential buyer enters into an existing real estate limited partnership, he or she must go essentially through the same selling process as the seller to sell the shares to a new party, including incurring the opportunity cost by being locked into the investment.

Empirically, we have evidence to suggest that the combined discount for lack of liquidity and lack of control for restricted stock transactions is around 13.5 to 50 percent of net asset value (see Gelman (1972), Trout (1977), Moroney (1973), and Maher (1976), among others). A few studies that analyze the primary offerings sold in a private placement find average discounts in the range of 41.7 to 46 percent, but with a much larger range of variability than for found in studies of restricted stock transactions (see Emory (2002), and Pratt (2000)). Work by Longstaff (2001) finds evidence that the average discount for lack of liquidity is around 15 to 80 percent of net asset value.

Most theoretical papers (which are quite copious) have rationalized the large discounts on minority-limited partnership interests with the notion that it is extremely costly to undertake a search for a buyer for a minority interest in a real estate limited partnership. The purpose of this paper is to address the question: Can search theory alone quantitatively explain the historically large discounts on minority-limited partnership interests in real estate limited partnerships and the time-variation thereof? To test the applicability of search theory to the valuation of minority-limited partnerships interests in a real estate limited partnership, I consider a simple variant of the model studied by Rocheteau and Weill (2011), which itself is a variant of Vayanas and Wang (2007) and Diamond (1982). The model allows me to analyze

how the level of buyer and seller search intensities leads to separate discounts for lack of liquidity and control. I then calibrate the model using parameters that are consistent with data on apartment buildings. Specifically, I select parameters that allow me to match the fixed asset supply of real estate limited partnerships with the number of minority shareholders. I then use these parameters along with an assessment of the flow of buyers who are likely to be matched with sellers to examine the quantitative implications of the model for the pricing of minority-limited partnership interests in real estate limited partnerships.

To preview the conclusions, I find that with realistic parameter values the model is able to generate discounts for lack of liquidity and lack of control that are consistent with the empirical literature. The comparative statics indicate that decreasing (increasing) the efficiency with which matches are made increases (decreases) the combined discount for lack of liquidity and lack of control. I also provide comparative statics that explore how holding costs incurred by minority shareholders as well as different levels of seller motivation affects the combined discount.

The paper is organized as follows. Section 2 presents the search model. Section 3 describes the calibration of the model parameters to the case of multifamily apartment buildings. Section 4 presents numerical simulations and comparative statics for the combined discount for lack of liquidity and lack of control. Section 5 compares the combined discounts computed from the model to those observed from data. Section 6 concludes the paper.

2 Rocheteau-Weill's Valuation Model

Consider the following simple variant of the Rocheteau and Weill (2011) search model. I consider a real estate limited partnership that owns a single apartment building that yields a constant cash-flow dividend, d. I assume here that the limited partnership has two types of partners: one or more general partners, of total mass, μ_m, and one or more minority partners, of mass $1-\mu_m$. I assume all general partners are each liable for all the debts of the partnership, while minority partners are liable for the debts of the partnership only to the

extent of the capital they have furnished. I further assume all general partners have an equal voice in the management of the property unless they otherwise agree, while the limited partners may not participate in any way in the management of the property.

I assume both general and minority shareholders are subject to liquidity shocks, for example a need for cash. I assume these shocks occur at a constant rate and only those investors who receive a liquidity shock – including both general and minority partners – are motivated to sell. However, when general partners want to sell, I assume they choose to exploit the property's distinct spatial advantages and sell the property in its entirety. In contrast, when minority investors want to sell, I assume that they are forced to sell their ownership interests in the secondary market, since they have no control whatsoever over the sale of the property.

Search frictions play a crucial role in this model. Buyers and sellers in the secondary market must search themselves out and agree to enter into a transaction. I assume that this search is costly, time consuming, and random. I add the assumption that buyers and sellers are atomistic and that buyers cannot become a general partner regardless of the number of minority shares they acquire. The latter is equivalent to assuming that strategic behavior in this market does matter.

The number of new buyers (restricted to those who are wealthy enough to be able to qualify as a sophisticated buyer) entering the secondary market is given by F. These investors are looking to buy out minority shareholders at a discount. I assume that buyers can only hold one minority unit. This assumption is innocuous, since minority interests in a limited partnership in all practicality cannot be converted into a majority stake regardless of the number of minority shares acquired.

Matching takes place via a Poisson process in which $\lambda M(\mu_b, \mu_s)$ is the number of matches per unit of time when there are μ_b buyers in the buyer state and μ_s sellers in the seller state. The parameter λ is a scale variable and M is equal to $\mu_b^{\varepsilon}\mu_s^{1-\varepsilon}$, where $0 < \varepsilon < 1$. The value of M is homogeneous of degree one, that is, increasing the levels of both μ_b and μ_s by the factor

a, increases the value of M by a. Further, the matching function M is equal to 0 when either $\mu_b = 0$ or $\mu_s = 0$, that is, when either side of the market has no agents. When matched, both the buyer and seller exit the market permanently. If no match occurs, the buyer and seller return to the market to be matched next period. After a period of time of random length, exponentially distributed with mean, $1/\kappa$, a buyer no longer wants to own a minority interest and thus exits the market (out of frustration) if no match occurs.

The probability a buyer meets a seller assuming the number of matches per unit of time is $\lambda M(\mu_b, \mu_s)$ is given by $\lambda_b = \lambda M(\mu_b, \mu_s)/\mu_b$. Similarly, the probability a seller meets a buyer in a period is given by $\lambda_s = \lambda M(\mu_b, \mu_s)/\mu_s$. The average time for a unit to be sold is $\mu_s/\lambda M(\mu_b, \mu_s)$, and the average time for a buyer to find a seller is $\mu_b/\lambda M(\mu_b, \mu_s)$.

The motion of the state variables, μ_b, μ_o, and μ_s, where μ_o denotes the measure of owners who are not seeking to sell, obey the ordinary differential equations

$$\dot{\mu}_b = F - (\lambda_b + \kappa)\mu_b \qquad (1)$$

$$\dot{\mu}_o = \lambda_b \mu_b - \kappa \mu_o \qquad (2)$$

$$\dot{\mu}_s = \kappa \mu_o - \lambda_s \mu_o \qquad (3)$$

and the equilibrium condition that the number of investors holding a minority interest, $\mu_o + \mu_s$, must equal the fixed supply of minority shares, S. An implication of (1) is that the measure of buyers, μ_b, increases with the flow of new buyers entering the secondary market, F, and decreases with the flow of buyers who are matched with sellers, $\lambda_b \mu_b$, and with the flow of buyers who exit the market, $\kappa \mu_b$. Equations (2) and (3) have a similar interpretation for the measure of owners, μ_o, and sellers, μ_s. The number of owners who are not seeking to sell, μ_o, increases with $\lambda_b \mu_b$ and decreases with flow of owners who are no longer seeking to sell and exit the market, $\kappa \mu_o$, while the number of sellers, μ_s, increases with $\kappa \mu_o$ (i.e., those who receive a preference shock) and decreases with the flow of sellers who are matched with buyers, $\lambda_s \mu_o$.

The general steady state solutions to (1)-(3) require $\mu_b = F/(\lambda_b + \kappa)$, $\mu_o = \lambda_b \mu_b / \kappa$, and $\mu_s = \lambda_b \mu_b / \lambda_s$. Substituting these values into the equilibrium condition that minority shares demands are equal to the fixed-share supplies yields

$$\left(\frac{F}{\kappa} - S\right)\lambda\left(\frac{1}{\theta}\right)^{1-\varepsilon} + \frac{F}{\theta} = \kappa S \qquad (4)$$

where $\theta = \mu_b/\mu_s$, $\lambda_b = \lambda\left(\frac{1}{\theta}\right)^{1-\varepsilon}$, and $\lambda_s = \lambda\theta^{\varepsilon}$. Equation (4) determines a unique ratio of buyers per seller, θ. With $\frac{F}{\kappa} > S$, the value of θ decreases with S and κ, and it increases with F and λ. As matching efficiency goes to infinity, $\lambda \to \infty$, the number of buyers per seller goes to infinity, $\theta \to \infty$. With $\lambda_s = \lambda\theta^{\varepsilon}$, the matching rate of sellers goes to infinity as well, $\lambda_s \to \infty$, thereby implying that sellers can sell their minority units instantly in the frictionless limit. As $\theta \to \infty$, the value of λ_b tends to the solution $\left(\frac{F}{\kappa} - S\right)\lambda_b = \kappa S$, where $\left(\frac{F}{\kappa} - S\right)$ is a measure of the number of investors who would be happy to own a minority interest, F/κ, minus the number of investors who actually own a minority interest, S. Thus, in the limit buyers search time adjusts so that it is aligned with the number of minority interests, κS, put up for sale.

I assume that the lifetime expected utility of buyers, owners, and sellers in the steady state solve the following Hamilton-Jacobi-Bellman equations:

$$rV_b = \lambda_b(-p + V_o - V_b) - \kappa V_b \quad (5)$$

$$rV_o = d + \kappa(V_s - V_o) \qquad (6)$$

$$rV_s = d - \gamma + \lambda_s(p - V_s) \qquad (7)$$

where r is the rate at which (risk-neutral) investors discount future utility, γ is the expected holding cost incurred by minority shareholders during an average sale, p is the equilibrium price of a minority share, V_o is the value of a minority share to an owner, V_b is the value of a minority share to a buyer, and V_s is the value of a minority share to a seller.

The interpretation of equations (5)-(7) is quite straightforward. From equation (5), a buyer finds a seller with Poisson arrival rate, λ_b. The minority shares are purchased at the price p and the buyer makes a transition to the owner state. The buyer's surplus from trade is $-p + V_o - V_b$, which is the monetary gain obtained by buyers because they are able to purchase a minority interest for a price, p, that is less than the highest price that they would be willing to pay, $V_o - V_b$. If no trade occurs and the buyer were to exit the market, the resulting reduction in utility is V_b. The latter occurs at the rate κ. From equation (6), owners receive a constant cash-flow dividend, d, per unit of time. On the other hand, owners transition to the seller state following a liquidity shock with Poisson arrival rate κ. In this event, the change in utility is $V_s - V_o$. From equation (7), sellers receive a constant cash-flow dividend, d, per unit of time, but incur a holding cost of γ. Here γ enters the picture to reflect the fact that sellers may have an immediate need for cash that cannot be delayed without serious harm or may face high financing costs, or may have a relative tax disadvantage. The main point here is that high values of γ lead to highly motivated sellers on the supply side. A seller finds a buyer with Poisson arrival rate, λ_s, and the seller exits the market. The seller's surplus from trade is $p - V_s$, which is the amount that sellers benefit by selling at a market price, p, that is higher than the least that they would be willing to sell for, V_s.

The linear system of equations (5)-(7) has the following solution:

$$p = \frac{d}{r} - \frac{\kappa\Sigma}{r} - \frac{\left(r + \lambda_b\right)\phi\Sigma}{r} \qquad (8)$$

where

$$\Sigma = \frac{\gamma}{r + \kappa + \lambda_b\phi + \lambda_s\left(1-\phi\right)} \qquad (9)$$

and where $\phi = \frac{-p + V_o - V_b}{\Sigma}$ and $1-\phi = \frac{p - V_s}{\Sigma}$ are the respective fractions of the match surplus appropriated by the buyer and seller.

To facilitate the interpretation of equation (8), it is useful to recognize that Σ is the total surplus of a match. From equation (5), the buyer's surplus from trade is $-p + V_o - V_b$, while from (7), the seller's surplus is $p - V_s$, which implies that the total surplus of a match is $\Sigma = V_o - V_b - V_s$. That is, the total surplus of a match is the value to an owner minus the values to a buyer and a seller. Substituting from equations (5)-(7) into $\Sigma = V_o - V_b - V_s$ and rearranging yields the expression in equation (9). From equation (9), the match surplus is the present value of the per-period holding costs discounted at the opportunity cost of selling of $r + \kappa + \lambda_b\phi + \lambda_s(1-\phi)$. This discount rate has three components: a rate of return of r which owners require to hold minority shares, a risk premium of κ to compensate for the κ percent chance that the sale breaks down and the potential match surplus vanishes, and a risk premium of $\lambda_b\phi + \lambda_s(1-\phi)$ to compensate for the inconvenience of performing a search. In the frictionless limit when either the buyer or the seller can trade without delays (i.e., when either $\lambda_b \to \infty$ or $\lambda_s \to \infty$), $\lambda_b\phi + \lambda_s(1-\phi) \to \infty$ and the match surplus, Σ, shrinks to zero.

Referring back to equation (8), which gives the expression for the equilibrium price of a minority share, three general points should be noted. First, the term $\frac{d}{r}$ is the present value of the constant cash-flow dividend. This term is equivalent to the price a buyer would pay for a majority interest. Here the discount rate r must reflect the risks that income flows could be adjusted downward and that vacancies and expenses could be revised upward. In addition, the discount rate r must also reflect an allowance for the illiquidity of the asset (i.e., the high financing or financial-distress costs associated with real estate, the relative size of the investor pool who are looking to invest in real estate, the efficiency of the matching process, the holding costs per time unit, etc.). The above needs to hold for there to be an equilibrium in the real estate market. The important feature of this result is that a gross rate of return of r is all owners need to achieve – including general partners – to hold the asset. Further, one could assume, for instance, that the incremental cost of holding a minority interest, over and above

the cost of holding the asset, γ, decreases as sellers trade just as easily as owners of the asset. As $\gamma \to 0$, $\Sigma \to 0$, causing the value of a minority interest to approach $\frac{d}{r}$.

Second, the term $\frac{\kappa\Sigma}{r}$ is a discount that compensates for illiquidity. This discount derives from the probability of receiving a liquidity shock, which occurs with intensity, κ, and the total surplus of a match, Σ. These two terms are multiplied together to calculate the expected cost of converting a minority interest in real estate limited partnership to cash. Indeed, because the limited partnership shares cannot be sold readily, the discount for lack of liquidity is related to Σ, which, in turn, is related to the expected search costs, γ, incurred by minority shareholders while locating a potential buyer. From (9), as γ increases, Σ increases. Other things equal, an increase in Σ implies that the term $\frac{\kappa\Sigma}{r}$ should increase. This discount is subtracted from the value of a majority interest, $\frac{d}{r}$, that is otherwise comparable but carries greater liquidity.

Third, the term $\frac{(r+\lambda_b)\phi\Sigma}{r}$ is a discount that corresponds to the lack of control inherent in a minority interest. Whereas the discount for illiquidity derives its value from the expected holding cost incurred by minority shareholders during an average sale, the discount for lack of control derives its value from the fraction of the bargaining discount, ϕ, which the buyer is able to extract from the seller in the secondary market. This discount can be avoided if minority-limited partners were to have control over the amount and timing of profit distributions. If the latter were the case, the minority-limited partner would then have the power to interfere in the conduct of the partnership business to force the sale of the property to achieve liquidity, thereby avoiding altogether the bargaining discounts that buyers are able to extract from the seller in the secondary market. As $\phi \to 1$, the discount for lack of control increases. But as $\phi \to 0$, the discount vanishes. The discount increases as λ_b increases, that

is, the discount increases as buyers are able to find motivated sellers and come to an agreement with these sellers to purchase their shares at a discount.

3 Calibration of Model Parameters

To evaluate the model described in the previous section, the model must be solved numerically using parameter values that roughly match the number of limited partnership shares traded per year, the average time to sell a limited partnership share, the cost of selling a limited partnership share, and the fraction of the bargaining discount that the buyer is able to extract from the seller in the secondary market. Parameter values must also be chosen to match the cash flow characteristics of the underlying property, including the cash-flow dividend yield and the rate at which investors discount future utility. Here I continue with my focus on multifamily residential apartment buildings, and thus ignore limited partnership interests in office, industrial, and retail shopping centers. The inclusion of limited partnership interests in office, industrial, and retail shopping centers would change very little in the analysis other than to provide some comparability to results presented below for multifamily residential apartments.

To obtain values that roughly match the number of limited partnership shares traded per year and the average time to sell a limited partnership share, I examine information on turnover rates for multifamily residential apartments in Chicago, Illinois from 1998 to 2013. The primary dataset for this analysis is based on a large database of housing transactions compiled by the Institute for Housing Studies (website: http://www.housingstudies.org/). For each transaction, the data contain the names of the buyer and seller, the transaction price, a property identification number, the transaction date, and numerous property characteristics, including number of units.

The data are quite comprehensive. The Institute for Housing Studies collects information from two primary sources. Its transaction information is from the Cook County Recorder of Deeds. Its property attribute information (number of units, square footage, year built, etc.) is

from the Cook County Assessor's office. The Institute for Housing Studies also collects information on owner type (sole proprietorship or limited partnership) from the Illinois Secretary of State.[1] I use the latter to determine the fraction of sales by limited partnerships by year.

Limited partnership occurrences and total trading volume are shown in Table 1 for the sample period 1998-2013. It will be seen at once that the peak years for limited partnership transactions (as a percentage of trading volume) are 2009-11. In 1998, the total number and value of limited partnership sales transactions were 3.5 and 25.5 percent of the total number and value of all transactions, respectively. In 2010, by contrast, the total number and value of limited partnership sales transactions were 61.1 and 90.2 percent of the total number and value of all transactions, respectively. The rise in the share of transactions involving a limited partnership arrangement represents in the main the influence of a decline in overall trading volume. Total trading volume peaked at $2.3 billion, on average, in 2005-07, up from $1.3 billion, on average, in 1998-2004. Since 2007, total trading volume is down 67 percent from its peak and down 41 percent from normal trading volume over 1998-2004.

Several additional points about the trading volumes in Table 1 are worth noting. First, transactions involving a limited partnership arrangement are generally a much larger percentage of value than number of transactions, a fact which indicates, as others have observed before (see Smith and Hess (2006)), that real estate limited partnerships are normally formed to take on large projects. A closer examination reveals that the average size of limited partnership projects is larger (over $2.6 million) compared to other projects (under $1.1 million). Second, limited partnerships are typically used when the owner of real estate wants reduce estate transfer costs. Valuation discounts for lack of liquidity and lack of control mean

[1] The Illinois Uniform Limited Partnership Act requires all limited partnerships in Illinois to obtain a Certificate of Limited Partnership. The certificate of limited partnership must include the name of the limited partnership, the address where records such as a partner list, tax filings and partnership agreements are kept, and the purpose of the limited partnership. If a limited partnership intends to be a limited liability limited partnership it must identify such election on the certificate of limited partnership. The name of the limited partnership must contain the words "Limited Partnership," "L.P.," "LP," or "LLLP," and cannot contain the words "Company," "Corporation," "Incorporated," "Inc.," "Co.," or "Corp." This naming convention is used by the Institute of Housing Studies to classify the sale as a sole proprietorship or limited partnership arrangement. No registration is required for a general partnership in Illinois. However, the Illinois Assumed Name Act does require the general partnership to register with its local county clerk's office when the general partnership's name is different from the owner(s) full legal name(s).

that limited partnership interest can be transferred at lower tax cost by using the gift tax annual exclusion to shelter the gifts or drawing down less of one's applicable unified credit amount. Another advantage of limited partnerships (especially in Illinois) is the avoidance of probate costs if an operating agreement provides how units are to otherwise pass at death. At death, the units are then allocated in accordance with the terms of the operating agreement, which avoids probate costs.

Using the estimates in Table 1, an attempt has been made here to get an approximate idea of the number of limited partners who might possibly want to sell their interests in the secondary market (or redeem their units in the case of a hedge fund). My procedure is to multiply the total number of transactions involving limited partnerships, expressed in percent of total outstanding limited partnerships, by $1/2$ (a minority or non-controlling interest is, by definition, less than 50 percent of outstanding units), and then to multiply the resultant value by the rate κ, i.e., the probability of receiving a liquidity shock. For example, if the turnover rate for limited partnerships holding multifamily residential apartments is equal to 0.03 and κ is between 0.05 and 0.10, then the number of minority limited partnership shares that are traded per year is between 0.00075 and 0.0015.

I set the value of the cash-flow dividend yield, d, equal to 0.054 (matching the 2012Q4 observation for current (appraised) value capitalization rate on multifamily apartments from a time-series of property capitalization rates published by NCREIF). I set the value of r, the rate at which investors discount future utility, equal to 0.105 (matching the average return on multifamily apartment buildings from 1978 to 2012 as published by NCREIF). I obtain a value for the equilibrium price of the property of $0.51(=0.054 \div 0.105)$, implying a steady state price-dividend ratio of 9.7.

The parameter λ is chosen with an eye towards Cheng, Lin, and Liu (2013). Cheng, Lin, and Liu (2013) point out that the average time-on-market for commercial real estate since 2008 is about 0.75 to 1.00 a year, or 9 to 12 months. The trading market for minority limited partnership shares is smaller and less liquid than commercial real estate assets. In the baseline

calibration, the average time-on-market for minority limited partnership shares, $1/\lambda$, is set equal to four times the average time-on-market for commercial real estate. This assumption is arbitrary, I grant. Accordingly, I also considered alternative cases (not reported here), in which this last assumption is replaced with a much shorter (longer) average time-on-market. None of the results described below is qualitatively altered by these alternative parameter values. Instead, the alternative values of $1/\lambda$ simply led to alternative parameter values for r, κ, λ_b, and λ_s. I assume γ, the expected holding costs incurred by minority shareholders during an average sale, is equal to 0.03. The remainder of the parameters can be solved in terms of these values (see below).

4 Numerical Discounts and Comparative Statics

In what follows, I present empirical estimates based on the model described in Section 2 of the discount for lack of liquidity (DLOL) and control (DLOC) for limited real estate partnership interests. I begin by fixing F at 30 and solving for θ, λ_b, and λ_s. Then, given values for λ_b, λ_s, κ, and F, I solve for μ_b and then μ_s, in that order (i.e., the number of buyers and sellers in the secondary market for minority limited partnership shares, respectively).

Given values of r, κ, λ_b, and λ_s, I use equation (9) to solve for Σ for various values of ϕ. Then, I use equation (8) to solve for the single, combined discount for lack of liquidity and lack of control for limited real estate partnership interests. I also examine comparative statics for the combined discount by varying various parameters from their base case values. For example, I explicitly allow the parameter values of γ, κ, θ, and λ to change.

All the parameters for the base case are listed in Table 2. These parameters use the assumed values for d, r, κ, γ, ε, and λ discussed in Section 3. Given these parameters, it turns out that the single, combined discount for lack of liquidity and lack of control is sizeable. Furthermore, in situations with faster/slower matching rates, the discount for lack of liquidity and lack of control behaves as expected, the discount increasing (decreasing) significantly with slower (faster) matching rates.

Table 3 reports the results of simulating the model over a range of parameter values. Notice that the value of ϕ ranges from a low 0 to a high 1.00 (i.e., from where the seller is able to achieve the highest possible price and end up with a surplus to where the buyer is able to extract 100 percent of the match surplus from the seller). To describe the comparative statics of the combined discount for lack of liquidity and lack of control, I allow the rate at which investors discount future utility, r, to vary between 5.5 and 15.5 percent. The range is intended to coincide with the underlying return requirements on core, core-plus, and value-added investment styles. In addition, to isolate the effects of γ, κ, θ, and λ, I let them vary one at a time. I allow the parameters γ and λ to decline and the parameters κ and θ to increase.

Reference to Table 3 establishes that the single, combined discount for lack of liquidity and lack of control in the base case varies from a low of 12.1 percent to a high of 64.8 percent as the parameter ϕ takes on values between 0 and 1.00. Other things equal, a 10 percent change in ϕ raises the combined discount by 2 to 9 percent. With $\phi = 1.0$ (i.e., with the entire match surplus appropriated by the buyer), the combined discount for lack of liquidity and lack of control varies between 55.6 and 64.8 percent.

Compared to the base case with $\phi = 0.5$, decreasing the expected holding costs incurred by minority shareholders during an average sale, γ, reduces the size of the combined discount for lack of liquidity and lack of control. It is important to realize that increasing the rate at which sellers exit the market if no match occurs, κ, increases the extent to which the seller is motivated to make a quick and costly sale. That explains why an increase in κ results in an increase in the combined discount for lack of liquidity and lack of control. Increasing the ratio of buyers per seller, θ, affects the combined discount for lack of liquidity and lack of control in predictable ways. Table 3 shows that as θ increases from 0.25 to 2.5 (a tenfold increase), the combined discount for lack of liquidity and lack of control decreases by about 20 percent on average as the market is definitely switching from a buyer's market to one that favor the seller more. Decreasing λ from the base case value increases the combined discount for lack of liquidity and lack of control as the matching efficiency decreases.

5 Discussion

The simulations discussed above suggest a number of propositions involving minority interests in real estate limited partnerships. The principal ones relate to: (a) the combined discount for lack of liquidity and lack of control can be quite large; (b) the combined discount is much larger when the rate at which investors discount future utility is larger; (c) as a larger fraction of the match surplus is appropriated by the buyer, the combined discount increases; as a larger fraction of the match surplus is appropriated by the seller, the combined discount will decrease; (d) the combined discount depends among other things on the cost of selling a limited partnership share; (e) the combined discount is somewhat greater when the seller is motivated to make a quick and costly sale; (f) the combined discount is somewhat smaller when the ratio of buyers per seller increases; and (g) if the average time to sell is larger, the combined discount is far greater.

It seems fitting, therefore, in this penultimate section, knowing what we know about the model, to ask, Is the model capable of, at the very least, replicating the price discounts that one observes in the data? Many efforts have been made to measure the price discount on minority interests in a limited partnership, and a fairly comprehensive summary of the relevant literature is provided in Table 4. The table describes each study with column headings for study, sample period, and observed average price discount.

Much of the existing literature can be classified into one of two approaches. The first involves analyzing discounts on sales of restricted shares of publicly traded companies. The second involves analyzing discounts on sales of closely held company shares compared to prices of subsequent initial public offerings of the same company's shares. In the first type publicly traded companies frequently issue a letter stock that is identical in all respects to a freely traded stock of a public company except it is restricted from trading on the open market for a certain period. In the second type funds are raised by private companies through securities which are sold not through a public offering, but rather through a private offering.

These securities sell at a discount relative to securities sold through a public offering because they have no established market in which they can readily be resold.

Table 4 is organized into two panels. Panel A presents the evidence from restricted shares literature. Panel B presents the evidence from the private transactions literature. The thirteen studies of restricted stock transactions reported in Table 4 are remarkably similar. All estimate the price difference between privately-placed letter-stock transactions and prices of the same but unrestricted securities eligible for trading on the open market. All find discounts for the lack of liquidity in the range of 13.5 to 50 percent. The Pittock and Stryker (1983) study finds a discount in the range from 7 to 91 percent, with a median price discount of 45 percent.

In contrast to the studies examining sales of restricted shares of publicly traded companies, the private transactions literature finds a discount for the lack of liquidity between 41.7 and 46 percent. The Emory (2002) study examines primary offerings sold in a private placement over the time period 1980 through 2000. Private placements do not trade in secondary markets. This lack of liquidity significantly reduces the prices that investors will pay for the issue. The Emory (2002) study does not control for market-wide stock movements. The Pratt (2000) study examines private transactions over the time period 1975 through 1997. The Pratt study compares prices per share of the public offering with "fair values" computed using various price multiples of price per share of the private transaction (which controls for differences in market conditions for stocks of the respective industries between the time of each private transaction and the time of each subsequent public offering). While the Pratt (2000) study finds an average differential between private transaction prices and public market prices of 41.7 percent, under certain conditions and over certain periods of time the discounts range from 24.2 to 55.6 percent.

Related literature (not shown in Table 4) solves the investor's intertemporal portfolio choice problem when the investor faces liquidity constraints (being unable to initiate or unwind a portfolio position instantly). This literature shows that illiquidity considerations

may lead risk-averse investors to select a portfolio very different from that which would be optimal if trading was unconstrained (Mayers (1972), Stapleton and Subrahmanyam (1979), Amihud and Mendelson (1986), Constantinides (1986), and Contantinides and Mehra (2002)). In addition, the risk of not being able to liquidate assets in a timely manner at a reasonable price may cause illiquid securities to be priced at large discounts relative to otherwise identical liquid securities. For example, Longstaff (2001), who modeled illiquidity in a stochastic volatility environment, and used trading strategies that were of bounded variation, finds a discount for the lack of liquidity in the range of 15 to 80 percent, depending on the current volatility of returns on the risky assets. Longstaff also finds that investors choose much lower initial portfolio weights for the risky asset in the presence of liquidity restrictions.

The above results suggest that discounts for the lack of liquidity and lack of control are substantial (higher than most observers seem to think they are). These discounts have features very similar to those generated by the model. Summarizing, the mean discounts are remarkably consistent over time and over a great variety of market conditions. Even so, some temporal variation does exist. Moreover, the discounts observed are generally higher in a more volatile price environment than in a less volatile state. One conclusion seems to be that there are potentially large gains from setting up a market for minority shares in real estate limited partnerships to improve liquidity.

6 Conclusions

This paper has constructed a model to value a minority interest in a real estate limited partnership and used it to evaluate minority partnership interests in multifamily properties. The specific model can explain remarkably well the average discount for illiquidity and lack of control observed in minority partnership interests in multifamily properties. The model is based on search theory and per-period holding costs, in which minority stakeholders looking to sell must find a buyer and every buyer must find an actual seller. All trading between the two groups takes place in a secondary market characterized by thin trading and illiquidity, and

the buying is done by a few buyers. The price offered by the buyers is generally lower than the true competitive price. The low price allows the buyers to capture a large share of the potential gains from trade.

The model is essentially the same as that of Rocheteau and Weill (2011). The model has three crucial terms. The first term is a term for the cash-flow dividend, capitalized at a discount factor that reflects the risks that income flows could be adjusted downward, the risk that vacancies and expenses could be revised upward, and an allowance for the illiquidity of the asset in the private real estate market. The second term is a term that represents the discount for lack of liquidity. This term depends on the probability of receiving a liquidity shock times the total surplus of a match, discounted at the rate at which investors discount future utility. The third term is a discount for lack of control. This term derives its value from the fraction of the bargaining discount that the buyer is able to extract from the seller in the secondary market. The latter can be avoided by controlling stakeholders, since they have the ability to put the jointly owned property up for sale in the private real estate market or not.

The model was originally set up to analyze the case of minority partnership interests in multifamily properties. The model is not limited, however, in its application to multifamily properties. The model can easily be applied to other property types as well, because the probability a buyer (seller) meets a seller (buyer), the rate at which investors discount future utility, and the total transaction costs incurred during an average sale all can be changed at will. A reasonable calibration of the model yields predicted discounts that are very close to what is observed in practice, i.e., from the vast empirical literature. Hence, I am led to conclude that search costs are able to explain the historically large discounts on minority-limited partnership interests in real estate limited partnerships and the time-variation thereof.

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Year	Total number of transactions	Total trading volume	Limited partnerships as percentage of total number of transactions	Limited partnerships as percentage of total trading volume
1998	1501	$979,098,375	3.5%	25.5%
1999	1648	$1,320,223,000	5.9%	35.3%
2000	1198	$1,109,030,580	10.6%	24.9%
2001	1101	$1,121,254,910	16.7%	32.7%
2002	1149	$1,108,420,900	20.0%	32.4%
2003	1146	$1,359,758,990	23.9%	38.5%
2004	1250	$1,988,075,400	33.2%	57.9%
2005	1146	$2,668,941,750	42.8%	59.2%
2006	896	$2,069,836,150	43.4%	56.0%
2007	665	$2,203,380,870	52.0%	48.8%
2008	381	$663,909,890	50.7%	73.3%
2009	314	$384,872,000	61.5%	83.2%
2010	306	$710,732,000	61.1%	90.2%
2011	422	$824,548,797	67.3%	87.0%
2012	512	$1,108,130,000	73.6%	79.8%
2013*	86	$66,321,500	69.8%	78.0%

Table 1. Total multifamily transactions volume and limited partnership occurrences, Cook County, IL, 1998-2013

* through 2013 first quarter.

Source: Multifamily transactions data are from the Cook County Office of Deed Recordings and cover every transaction in Cook County over the period 1998-2013. These data have been collected by the DePaul Institute for Housing Studies. For each transaction, the data contain the names of the buyer and seller, the transaction price, the address, a property identification number, and the transaction date. The multifamily transactions data are matched to legal ownership data from the Office of the Illinois Secretary of State using a property identification number. The legal ownership data from the Office of the Illinois Secretary of State does not provide direct information regarding the organization form beyond stating whether a firm is privately held or publicly owned. Hence, to identify limited partnership occurrences I rely on the legal rules governing entity names. For example, an entity whose name ends with the abbreviation "L.L.P." is treated as a limited liability partnership. The data above refer to the existence of limited liability partnerships.

Variable	Value	Definition
d	0.054	cash flow dividend (as a % of price)
r	0.105	required return
κ	0.05	match rate (exits after $1/\kappa$ periods if no match occurs)
γ	0.03	expected holding costs incurred by minority shareholders during an average sale
ε	0.5	$0 < \varepsilon < 1$
λ	0.25	scaling parameter ($1/\lambda$ is the average time to sell)
F	30	fixed supply of buyers (10% of total limited partnership transactions)
θ	0.25	$= \mu_B/\mu_S$, i.e., equilibrium ratio of buyers to sellers
λ_B	0.50	$= \lambda \times (1/\theta)^{1-\varepsilon}$
λ_S	0.125	$= \lambda \times \theta^{\varepsilon}$
μ_B	54.5	$= F/(\lambda_B + \kappa)$, i.e., number of buyers in market
μ_S	218.2	$= \lambda_B \times \mu_B/\lambda_S$, i.e., number of sellers in market
$V_0 - V_B$	0.527	expected utility of majority owner - value of minority share to buyer
V_S	0.469	value of minority share to seller
$\lambda \times M$	27.3	$= \lambda \times \mu_B^{\varepsilon} \times \mu_S^{1-\varepsilon}$, i.e., number of matches
ϕ	0.75	fraction of surplus appropriated by buyer
$1 - \phi$	0.25	fraction of surplus appropriated by seller
Σ	0.059	$= V_o - V_b - V_s$

Table 2. Parameter values for base case

						Comparative statics $\phi = 0.50$			
r	$\phi = 0$	$\phi = 0.25$	$\phi = 0.50$	$\phi = 0.75$	$\phi = 1.0$	$\gamma = 0.015$	$\kappa = 0.50$	$\theta = 2.5$	$\lambda = 0.025$
0.055	12.1%	32.5%	43.7%	50.8%	55.6%	21.8%	49.8%	22.8%	42.1%
0.065	12.1%	32.9%	44.4%	51.6%	56.6%	22.2%	50.2%	23.6%	44.2%
0.075	12.1%	33.4%	45.0%	52.4%	57.5%	22.5%	50.5%	24.3%	46.2%
0.085	12.1%	33.8%	45.7%	53.2%	58.4%	22.8%	50.8%	25.0%	48.3%
0.095	12.1%	34.2%	46.4%	54.0%	59.3%	23.2%	51.1%	25.8%	50.3%
0.105	12.1%	34.6%	47.0%	54.8%	60.2%	23.5%	51.5%	26.5%	52.4%
0.115	12.1%	35.1%	47.7%	55.7%	61.2%	23.8%	51.8%	27.2%	54.4%
0.125	12.1%	35.5%	48.4%	56.5%	62.1%	24.2%	52.1%	28.0%	56.5%
0.135	12.1%	35.9%	49.0%	57.3%	63.0%	24.5%	52.4%	28.7%	58.5%
0.145	12.1%	36.4%	49.7%	58.1%	63.9%	24.8%	52.7%	29.4%	60.6%
0.155	12.1%	36.8%	50.4%	58.9%	64.8%	25.2%	53.1%	30.1%	62.6%

Table 3. Combined discounts for lack of liquidity and lack control for minority interests in real estate limited partnerships, %

Definitions of variable symbols: r = the rate at which (risk-neutral) investors discount future utility. ϕ = fraction of the match surplus appropriated by the buyer. γ = the expected holding cost incurred by minority shareholders during an average sale. κ = the match rate (seller exits after $1/\kappa$ periods if no match occurs). θ = the equilibrium ratio of buyers to sellers. λ = a scaling parameter ($1/\lambda$ is the average time to sell).

Study	Sample period	Observed average price discount
A. Data Used: Restricted Stocks		
Gelman (1972)	1968-70	33.0%
Trout (1977)	1968-72	33.5%
Moroney (1973)	1969-72	35.6%
Maher (1976)	1968-73	35.4%
Pittock & Stryker (1983)	1978-82	45.0%
Hertzel & Smith (1993)	1980-87	13.5%
Silber (1991)	1981-88	33.8%
Baja, Denis, Ferris, and Sarin (2001)	1990-95	22.2%
Johnson (1999)	1991-95	20.0%
Oliver & Meyers (2000)	1980-96	27.0%
Robak & Hall (2001)	1980-97	50.0%
Aschwald (2000)	1996-97	21.0%
Robak (2007)	2005-06	32.8%
B. Data Used: Private Transactions		
Emory (2002)	1980-2000	46.0%
Pratt (2000)[a]	1975-97	41.7%

Table 4. Review of the empirical literature

Notes to Table 4: Studies are grouped according to the approach that was employed by them. One approach has been to analyze discounts on sales of restricted shares of publicly traded companies (i.e., restricted stocks). The second approach has been to analyze discount on sales of closely held company shares compared to prices of subsequent initial public offerings of the same company's shares (i.e., private transactions). A third approach (not listed) has been to model illiquidity in a stochastic volatility environment, allowing investors to choose much lower initial portfolio weights for the risky asset in the presence of liquidity restrictions.

[a] Otherwise known as the Willamette Management Associates.